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ScienceDirect
Pattern Recognition Letters 28 (2007) 278–285
Pattern Recognition Letters
www.elsevier.com/locate/patrec

Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis

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Received 2 December 2004; received in revised form 10 May 2006
Available online 7 September 2006

Communicated by A.M. Alimi

Abstract

Existing similarity measures between intuitionistic fuzzy sets/vague sets are analyzed, compared and summarized by their counter-intuitive examples in pattern recognition. The positive aspects of each similarity measure are demonstrated, along with counter cases and discussion of the conditions under which each may not work as desired. The research presented here could benefit selection and applications of similarity measures for intuitionistic fuzzy sets and vague sets in practice.

Keywords: Pattern recognition; Similarity measures; Intuitionistic fuzzy sets (IFSs); Vague sets

1. Introduction

Fuzzy sets theory, proposed by Zadeh (1965), is a realistic and practical means to describe the objective world that we live. The method has successfully been applied in various fields. As a generalization of fuzzy sets, intuitionistic fuzzy sets (IFSs) was presented by Atanassov (1986), and vague sets were proposed by Gau and Buehrer (1993). Bustince and Burillo (1996) pointed out that the notion of vague sets was the same as that of IFSs. Atanassov published a number of additional studies (Atanassov, 1989, 1994a,b, 1995, 1999). IFSs/vague sets make descriptions of the objective world more realistic, practical, and accurate, making it very promising. They have been widely applied in decision making (Szmidt and Kacprzyk, 1996), logic programming (Atanassov and Gargov, 1990; Atanassov and Georgeiv, 1993) medical diagnosis (De et al., 2001), pattern recognition (Hung and Yang, 2004) and seems to have been more popular than fuzzy sets technology in recent years.

A similarity measure is used for estimating the degree of similarity between two sets. Since Zadeh proposed fuzzy sets, many scholars have conducted research on similarity measures between fuzzy sets from all kinds of viewpoints, which have been or could be applied in areas such as data preprocessing, for identifying the functional dependency relationships between concepts in data mining systems, for approximate reasoning (Li et al., 2002; Tianjiang et al., 2002), and for other purposes to include pattern recognition (Dengfeng and Chuntian, 2002; Mitchell, 2003; Zhizhen and Pengfei, 2003 and Hung and Yang, 2004). Other similarity measures proposed recently for IFSs as a generalization of fuzzy sets include Chen (1995, 1997), Hong and Kim (1999) and Fan and Zhangyan (2001).

In this paper, existing similarity measures between IFSs/vague sets are analyzed, compared and summarized by their counter-intuitive examples in pattern recognition. The focus of this study is on apparent weaknesses of these similarity measures, and the conditions or reasons they do

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doi:10.1016/j.patrec.2006.07.009

not work. This research could benefit selection and applications of similarity measures between IFSs/vague sets in practice.

2. Preliminaries

This section reviews basic definitions and terms. Some principles involved in this paper are as follows:

2.1. Intuitionistic fuzzy sets (IFSs)

This section describes the basic definitions relating to fuzzy sets.

Definition 2.1.1 (*Fuzzy Sets*). Zadeh (1965) defined a Fuzzy Set A in the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ as follows:

$$A = \{(x, \mu_A(x)) | x \in X, \mu_A(x) \in [0, 1]\}$$

$X \to u_A(x)$ is called membership function of A, $u_A(x)$ indicates the membership degree of x to A, $u_A(x)$ takes its value on interval $[0,1]$. The bigger the value of $u_A(x)$ is, the greater the degree of membership of x to A is. $u_A(x)$ indicates the proofs of both pros and cons. It is impossible for $u_A(x)$ to denote only pros or only cons or at the same time both of them. This is a problem solved in part by IFSs.

Definition 2.1.2 (*Intuitionistic fuzzy sets* (*IFSs*)). Atanassov (1986) gave an IFS V in the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ defined as follows:

$$V = \{(x, t_v(x), f_v(x)) | x \in X, t_v(x) \in [0, 1], f_v(x) \in [0, 1], \\ 0 \leqslant t_v(x) + f_v(x) \leqslant 1\}$$

$t_v(x)$ and $f_v(x)$ denotes a membership function and non-membership function of x to V separately. $t_v(x)$ is the lowest bound of membership degree derived from proofs of supporting x; $f_v(x)$ is the lowest bound of non-membership degree derived from proofs of rejecting x, It is clear that the membership degree of IFS V has been restricted in $[t_v, 1 - f_v]$, which is a subinterval of $[0,1]$.

Let $h_v(x)$ denote $1 - t_v(x) - f_v(x)$, which could be regarded as a degree of uncertainty or hesitancy of x to V or waver, if $t_v(x) = 1 - f_v(x)$, implying that we know x precisely. If so, IFSs degenerate into fuzzy sets; If $t_v(x) = 0$ and $f_v(x) = 1$, or, $t_v(x) = 1$ and $f_v(x)$=0, indicating the information on x is very precise, IFSs degenerate into crisp sets. Thus we can conclude that IFSs are extensions of fuzzy sets and crisp sets.

Definition 2.1.3. If an IFS $V = \Phi$, iff $t_v \equiv 0$ and $f_v \equiv 1$.

Definition 2.1.4. Complement set $\overline{V}$of an IFS V is defined as $t_{\overline{V}} = f_V$, $1 - f_{\overline{V}} = 1 - t_V$.

Definition 2.1.5. A and B denote an IFS respectively, If $A = B$, iff $t_A = t_B$, $1 - f_A = 1 - f_B$.

Definition 2.1.6. $A \subseteq B$ iff $t_A \leqslant t_B$ and $f_A \geqslant f_B$.

2.2. Similarity measure between IFSs

Dengfeng and Chuntian (2002) introduced the following definition of similarity measure between IFSs:

Definition 2.2.1. A mapping $S : \mathrm{IFSs}(X) \times \mathrm{IFSs}(X) \to [0, 1]$. $\mathrm{IFSs}(X)$ denotes the set of all IFSs in $X = \{x_1, x_2, \ldots, x_n\}$. $S(A, B)$ is said to be the degree of similarity between $A \in \mathrm{IFSs}(X)$ and $B \in \mathrm{IFSs}(X)$, if $S(A, B)$ satisfies the properties condition – (P1–P5)

P1: $S(A, B) \in [0, 1]$,
P2: $S(A, B) = 1 \Longleftrightarrow A = B$,
P4: $S(A, B) = S(B, A)$,
P4: $S(A, C) \leqslant S(A, B)$ and $S(A, C) \leqslant S(B, C)$ if $A \subseteq B \subseteq C$, $C \in \mathrm{IFSs}(X)$,
P5: $S(A, B) = 0 \Longleftrightarrow A = \Phi$ and $B = \overline{A}$, or, $A = \overline{B}$ and $B = \Phi$.

Remark. P2 is a new 'strong' version proposed by Hung and Yang (2004), but has been taken for granted by Chen (1995, 1997), Hong and Kim (1999), Fan and Zhangyan (2001) and Li et al. (2002). P5 is assumed in these five papers (which concern vague sets), but not by Dengfeng and Chuntian (2002), Mitchell (2003), Zhizhen and Pengfei (2003), or by Hung and Yang (2004). These last four papers address IFSs. Bustince and Burillo (1996) have pointed out the notion of vague sets is the same as that of IFSs. After consideration, we think the addition of P5 is necessary, which makes the definition of similarity measure between IFSs more strict and precise. In the next section some counter-intuitive cases are shown to result from similarity measures that do not satisfy the property condition P5.

3. Analysis on existing similarity measure between IFSs/vague sets

In this section, comprehensive analysis of similarity measures between IFSs/vague sets are provided. First, let $S(A, B)$ be similarity measure between $A \in \mathrm{IFSs}(X)$ and $B \in \mathrm{IFSs}(X)$. The meaning of all the signs in the following are same as those in the last section.

Chen (1995, 1997) proposed the concept of similarity measures between vague sets and defined its expression $S_C(A, B)$ as follows:

$$S_C(A, B) = 1 - \frac{\sum_{i=1}^{n} |S_A(x_i) - S_B(x_i)|}{2n} \tag{1}$$

Here $S_A(x_i) = t_A(x_i) - f_A(x_i)$ and $S_B(x_i) = t_B(x_i) - f_B(x_i)$, called Core of A and B or degree of support of A and B respectively, $S_A(x_i) \in [-1,1]$, $S_B(x_i) \in [-1,1]$. Chen believes that the bigger $S_C(A,B)$ is, the more similar A and B are, but when $A = \{(x,0,0)\}$ and $B = \{(x,0.5,0.5)\}$, we get $S_C(A,B) = 1$ according to formula (1), indicating A is same as B, it is obviously counter-intuitive, which both Hong and Kim (1999) and Fan and Zhangyan (2001) mentioned. $S_C(A,B)$ concerns only the degree of support. For $S_C(A,B)$, we have $t_A - f_A = t_B - f_B \Rightarrow S_C(A,B) \equiv 1$, that is to say if degree of support of A is equal to degree of support of B, then A and B will be the same, which leads to a series of cases satisfying $t_A - f_A = t_B - f_B$ and then $S_C(A,B) = 1$, all of which are counter-intuitive except the case of $A = B$. $S_C(A,B)$ is too rough a measure of the degree of similarity. We found that the existence of counter-intuitive cases results from the circumstance that $S_C(A,B)$ does not satisfy property condition P2.

Hong and Kim (1999) and Fan and Zhangyan (2001) proposed new similarity measures S_H and S_L as follows. Together these definitions could overcome the problem occurring in S_C

$$S_H(A,B) = 1 - \frac{\sum_{i=1}^{n}(|t_A(x_i) - t_B(x_i)| + |f_A(x_i) - f_B(x_i)|)}{2n} \quad (2)$$

$$S_L(A,B) = 1 - \frac{\sum_{i=1}^{n}|S_A(x_i) - S_B(x_i)|}{4n} - \frac{\sum_{i=1}^{n}|t_A(x_i) - t_B(x_i)| + |f_A(x_i) - f_B(x_i)|}{4n} \quad (3)$$

S_H focuses on the difference between t_A and t_B as well as the difference between f_A and f_B. Supposing A, B, C and D are vague sets. For S_H, if $|t_A(x_i) - t_B(x_i)| = |t_C(x_i) - t_D(x_i)|$ and $|f_A(x_i) - f_B(x_i)| = |f_C(x_i) - f_D(x_i)|$, then $S_H(A,B) = S_H(C,D)$, because of the existence of sign of absolute value, for $|t_A(x_i) - t_B(x_i)|$, $|t_C(x_i) - t_D(x_i)|$, $|f_A(x_i) - f_B(x_i)|$ or $|f_C(x_i) - f_D(x_i)|$, S_H cannot distinguish positive difference from negative difference, so there are still counter cases (type I) for S_H as follows: Supposing $A = \{(x,0.3,0.3)\}$, $B = \{(x,0.4,0.4)\}$, $C = \{(x,0.3,0.4)\}$ and $D = \{(x,0.4,0.3)\}$. According to formula (2), we have $S_H(A,B) = S_H(C,D) = 0.9$, which is not intuitively consistent. In addition, when

$$\begin{aligned} &|t_A(x_i) - t_B(x_i)| + |f_A(x_i) - f_B(x_i)| \\ &\quad = |t_C(x_i) - t_D(x_i)| + |f_C(x_i) - f_D(x_i)|, \\ &S_H(A,B) = S_H(C,D) \end{aligned}$$

occurs, another type of counter-intuitive case (type II) occurs where $A = \{(x,1,0)\}$, $B = \{(x,0,0)\}$ and $C = \{(x,0.5,0.5)\}$, $S_H(A,B) = S_H(C,B) = 0.5$.

S_L has inherited strengths from S_C and S_H, overcomes their counter cases concerning both degree of support and the differences between t_A and t_B as well as that between f_A and f_B. S_L, compared with S_H, displays preference to $t_A \leqslant t_B$, $1 - f_A \geqslant 1 - f_B$ case under the condition of the same difference between membership degrees as well as the same difference of non-membership degrees, which actually enhances the distinguish ability between positive difference and negative difference between membership or between non-membership degrees, better than S_C and S_H, more consistent with intuition, but still cannot avoid limitations of S_C and S_H completely. If $|S_A(x_i) - S_B(x_i)| + |t_A(x_i) - t_B(x_i)| + |f_A(x_i) - f_B(x_i)| = |S_C(x_i) - S_D(x_i)| + |t_C(x_i) - t_D(x_i)| + |f_C(x_i) - f_D(x_i)|$, then $S_L(A,B) = S_L(C,D)$, which leads to the counter-intuitive case such as when $A = \{(x,0.4,0.2)\}$, $B = \{(x,0.5,0.3)\}$ and $C = \{(x,0.5,0.2)\}$, $S_L(A,B) = S_{LH}(A,C) = 0.95$, which does not seem reasonable.

Yanhong et al. (2002) proposed a new similarity measure S_O as follows:

$$\begin{aligned} S_O(A,B) &= 1 - \sqrt{\frac{\sum_{i-1}^{n}(t_A(x_i) - t_B(x_i))^2 + (f_A(x_i) - f_B(x_i))^2}{2n}} \\ &= 1 - \sqrt{\frac{\sum_{i-1}^{n}[(t_A(x_i) - f_A(x_i))^2 + (t_B(x_i) - f_B(x_i))^2 - 2(t_B(x_i) - f_A(x_i))(t_A(x_i) - f_B(x_i))]}{2n}} \end{aligned} \quad (4)$$

S_O emphasizes the degree of support, the difference between t_A and t_B, and the difference between f_A and f_B, S_O could avoid the counter-intuitive cases of S_C, type II of S_H and S_L, but has the same counter-intuitive cases as type I of S_H (see Appendix 1), which results from the same reason as S_H.

Dengfeng and Chuntian (2002) proposed their similarity measure of IFSs, which we call S_{DC}. They applied this measure to pattern recognition. This measure was originally presented as a form of weighted similarity measure. For comparability, we change it into normal form (i.e. every element is of the same importance), and thus S_{DC} is

$$S_{DC}(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}|\psi_A(x_i) - \psi_B(x_i)|^p}{n}} \quad (5)$$

Here p is a parameter,

$$\psi_A(x_i) = \frac{t_A(x_i) + 1 - f_A(x_i)}{2}, \quad \psi_B(x_i) = \frac{t_B(x_i) + 1 - f_B(x_i)}{2}$$

In fact, Dengfeng and Chuntian first converted A and B into ordinary fuzzy sets $\psi_A(x_i)$ and $\psi_B(x_i)$, then applied Minkowski distance to calculate similarity degree of fuzzy sets. When $p=1$, $S_{DC}=S_C$. No matter what value p takes, S_{DC} has the same type of counter-intuitive cases as S_C. $\psi_A(x_i)$ and $\psi_B(x_i)$ are actually median values of interval $[t_A(x_i), 1-f_A(x_i)]$ and $[t_B(x_i), 1-f_B(x_i)]$ respectively. We could also explain the con cases of S_{DC} as follows: if median values of two intervals for IFSs are equal to each other, then $S_{DC}=1$, and thus there will be a lot of counter-intuitive cases, and S_{DC} measures similarity a bit roughly. Mitchell (2003) presented other counter-intuitive cases for S_{DC}, similar in nature to those of S_C. The existence of counter-intuitive cases results from S_{DC} not satisfying property condition – P2 strictly, which is the same as with S_C.

Mitchell (2003) gave a simple modification of S_{DC} and corrected S_{DC}'s problem. He adopted a statistical viewpoint and interpreted A and B as ensembles of ordered membership functions which fill the space between $t_A(x_i)$ and $1-f_A(x_i)$ as well as between $t_B(x_i)$ and $1-f_B(x_i)$. Let $\rho_t(A,B)$ and $\rho_f(A,B)$ denote, respectively, the similarity measures between the low membership function $t_A(x_i)$ and $t_B(x_i)$ as well as between the high membership function $1-f_A(x_i)$ and $1-f_B(x_i)$:

$$\rho_t(A,B) = S_{DC}(t_A(x_i), t_B(x_i)),$$
$$\rho_f(A,B) = S_{DC}(1-f_A(x_i), 1-f_B(x_i)).$$

Then, the modified S_{DC}, called S_{HB}, is as follows:

$$S_{HB}(A,B) = \frac{1}{2}(\rho_t(A,B) + \rho_f(A,B)) \tag{6}$$

In fact, $S_{HB}(A,B)$ (when $p=1$ or for one-element set) $= S_H(A,B)$, so S_{HB} has the same two types of counter cases as S_H (see Appendix 1). For the same reason, S_{HB} has the same counter cases as S_H.

To overcome the weakness of S_{DC}, Zhizhen and Pengfei (2003) proposed $S_e^p(A,B)$, $S_s^p(A,B)$ and $S_h^p(A,B)$ as follows:

$$S_e^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}(\phi_t(x_i) + \phi_f(x_i))^p}{n}} \tag{7}$$

Here $\phi_t(x_i) = |t_A(x_i) - t_B(x_i)|/2$, $\phi_f(x_i) = |(1-f_A(x_i))/2 - (1-f_B(x_i))/2|$. This measure makes use of two end points of the subinterval in IFSs to define similarity measures, focusing on the difference between t_A and t_B as well as difference between f_A and f_B. When $p=1$ or for one-element set, $S_e^p(A,B) = S_{HB}(A,B)$, $S_e^p(A,B) = S_H(A,B)$. So S_e^p has same type of counter-intuitive cases as S_{HB} and S_H.

$$S_s^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}(\varphi_{s1}(x_i) + \varphi_{s2}(x_i))^p}{n}} \tag{8}$$

$$\varphi_{s1}(x_i) = |m_{A1}(x_i) - m_{B1}(x_i)|/2$$
$$\varphi_{s2}(x_i) = |m_{A2}(x_i) - m_{B2}(x_i)|/2$$
$$m_{A1}(x_i) = (t_A(x_i) + m_A(x_i))/2$$
$$m_{B1}(x_i) = (t_B(x_i) + m_B(x_i))/2$$
$$m_{A2}(x_i) = (m_A(x_i) + 1 - f_A(x_i))/2$$
$$m_{B2}(x_i) = (m_B(x_i) + 1 - f_B(x_i))/2$$
$$m_A(x_i) = (t_A(x_i) + 1 - f_A(x_i))/2$$
$$m_B(x_i) = (t_B(x_i) + 1 - f_B(x_i))/2$$

For the interval $[t_A(x_i), 1-f_A(x_i)]$ in A, $m_A(x_i)$ is the median value of the interval. In this case, the interval is divided into two subintervals, denoted as $[(t_A(x_i), m_A(x_i))]$ and $[m_A(x_i), 1-f_A(x_i)]$, $m_{A1}(x_i)$ and $m_{A2}(x_i)$ are median values of the two subintervals separately, so do $m_B(x_i)$, $m_{B1}(x_i)$ and $m_{B2}(x_i)$. So the formula (8) could avoid the problem of each interval having equal median values, but S_{DC} has this problem. For S_s^p, when $A=\{(x, 0.4, 0.2)\}$, $B=\{(x, 0.5, 0.3)\}$ and $C=\{(x, 0.5, 0.2)\}$, $S_s^p(A,B) = S_s^p(A,C) = 0.95$, it does not seem reasonable, and is same as for $S_L(A,B)$.

S_h^p makes full use of known information on IFSs such as length of subinterval and the median value of subinterval, we show the form without weight of $S_h^p(A,B)$ for comparability as follows:

$$S_h^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}(\eta_1(i) + \eta_2(i) + \eta_3(i))^p}{3n}} \tag{9}$$

$\eta_1(i) = \phi_t(x_i) + \phi_f(x_i)$ (occurring in S_e^p) or $\eta_1(i) = \varphi_{s1}(x_i) + \varphi_{s2}(x_i)$ (occurring in S_s^p),
$\eta_2(i) = \psi_A(x_i) - \psi_B(x_i)$ (occurring in S_{DC}),
$\eta_3(i) = \max(l_A(i), l_B(i)) - \min(l_A(i), l_B(i))$

denotes dissimilarity degree of the length, among which $l_A(i) = (1 - f_A(x_i) - t_A(x_i))/2$, $l_B(i) = (1 - f_B(x_i) - t_B(x_i))/2$. Which is relative powerful methods for considering more information to measure the similarity degree of IFSs, S_s^p can avoid all the cons cases that the above similarity measures have.

Hung and Yang (2004) presented three new similarity measures between IFSs based on Hausdorff distance:

$$S_{HY}^1(A,B) = 1 - d_H(A,B) \tag{10}$$
$$S_{HY}^2(A,B) = (e^{-d_H(A,B)} - e^{-1})/(1 - e^{-1}) \tag{11}$$
$$S_{HY}^3(A,B) = (1 - d_H(A,B))/(1 + d_H(A,B)) \tag{12}$$

Here $d_H(A,B) = \frac{1}{n}\sum_{i=1}^{n}\max(|t_A(x_i) - t_B(x_i)|, |f_A(x_i) - f_B(x_i)|)$. For S_{HY}^1, S_{HY}^2 and S_{HY}^3, all of them face the counter-intuitive cases of S_L and type I of S_H. For S_{HY}^1, S_{HY}^2 and S_{HY}^3, based on one-element IFS, if $\max(|t_A(x_i) - t_B(x_i)|, |f_A(x_i) - f_B(x_i)|) = \max(|t_C(x_i) - t_D(x_i)|, |f_C(x_i) - f_D(x_i)|)$, then $S_{HY}(A,B) = S_{HY}(C,D)$. It is obvious that a lot of cases satisfy these conditions, so the S_{HY} series measure similarity too roughly, leading to counter-intuitive cases. Moreover, S_{HY}^1, S_{HY}^2 and S_{HY}^3 admit $S_{HY}(A,B) = 0$ when $A = [(x,1,0)]$ and $B = [(x,0,0)]$, which results from that S_{HY}^1, S_{HY}^2 and S_{HY}^3 disobeying similarity measure property condition – P5.

4. Example for selection procedure

We demonstrate the selection procedure of similarity measures by synthetic data in a hiring decision. The problem is involved in 3 applicants (Antonio, Fabio, Alberto) for a position, each is evaluated over four attributes, which are experience in the specific job function (A_1), Educational background (A_2), adaptability (A_3), and aptitude for teamwork (A_4).

We obtain the decision matrix A as follows

$$A = \begin{array}{c} \\ \text{Antonio} \\ \text{Fabio} \\ \text{Alberto} \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} [0.3, 0.6] & [0.5, 0.5] & [0, 1] & [0.4, 0.8] \\ [0.3, 0.7] & [0.3, 0.4] & [1, 1] & [0.3, 0.4] \\ [0.4, 0.7] & [0, 0.5] & [0.5, 0.5] & [0.5, 0.8] \end{bmatrix} \end{array}$$

We use the ideas of grey relation analysis presented by Julong (1982) for choosing the best candidate for the position, but when calculating the similarity, replace grey relational coefficient $r(X_O(k), X_i(k))$ between the reference sequence X_O and the alternate sequence X_i at point k using some similarities measure method. r_i denotes the degree of similarity between the ith alternative X_i and reference sequence X_0. The optimal alternative $r_t = \max_{1 \leqslant i \leqslant m} r_i$.

The paper takes the weight of each attribute as 1. The reference sequence X_0 as follows is composed of the optimal interval value of indicator over all alternatives

$$X_0 = [\,[0.4, 0.7] \quad [0.5, 0.5] \quad [1, 1] \quad [0.5, 0.8]\,]$$

Among the existing similarity measures, We would like to choose among S_C, S_{DC}, S_H, S_L and S_h^p for they focus on different difference as showed in Appendix 3, just using notation here, that is to say, S_C focuses on C3, S_{DC} focuses on C2, S_H focuses on C1, S_L focuses on C1 and C3, and $S_h^p(A, B)$ focuses on C1, C2 and C3.

If you focus on waver, perhaps some VIP did not show their support or reject clearly, then S_h^p, which considers $1 - t_A(x_i) - f_A(x_i)$, might be the best. If you feel $|t_A(x_i) - f_A(x_i)|$ should be the primary focus, in which case you might choose S_C or S_L etc. But it is not enough just based above consideration, at the same time, we should pay attention to characteristics of our data, we have to check if there lie data types that some similarity measure does not work. in our data. In fact, our data include some data types that S_C, S_{DC}, S_H and S_L does not work. The detailed explanation is omitted here. So $S_h^p(A, B)$ is our best solution.

For comparison, we show the final results of the above five similarity measures as in Table 1.

You can see, by S_C, S_{DC}, S_H, S_L, the optimal alternative is Antonio, but by $S_h^p(A, B)$ approach, the optimal alternative $r_t = \max_{1 \leqslant i \leqslant m} r_i = r_3$, that is Alberto. There lie different results based on different solutions. In a word, the selection procedure is very necessary and important.

Table 1
The optimal alternate based on five different similarity measures

	r_1	r_2	r_3	r_t
$S_C(A, B)$	0.838	0.813	0.813	0.838
$S_{DC}(A, B)$	0.838	0.813	0.813	0.838
$S_H(A, B)$	0.838	0.813	0.813	0.838
$S_L(A, B)$	0.838	0.813	0.813	0.838
$S_h^p(A, B)$	0.846	0.838	0.854	0.854

5. Conclusion

This paper compared and analyzed much of the research published on similarity measures between IFSs/vague sets. Although the comparison is based on single-element sets, it is the basis of that of the multi-element sets (the counter cases of multi-element sets can be constructed according to that of one-element counter cases or sets) and can demonstrate drawbacks of some similarity measures accurately and thoroughly. Analysis results based on multi-element sets are apt to hide shortcomings because of complexity. At the same time, comparisons based on single-element sets are simple, direct, and easy to understand. Based on the comparison and analysis in last section, we provided summary information of similarity measures as Appendices 1–3.

The distinguishability of a similarity measure is determined by expression form and the information that expression contains or focuses on. Based on Appendices 1 and 2, we know the more the information that the similarity measures focus on, the more powerful their distinguishability. There is a lot of information applied to discern similarity between IFSs/vague sets by existing similarity measures, including difference of degree of support, difference of t_A and t_B, difference of f_A and f_B, median values of intervals or subintervals and length of interval. As to which similarity measures focus on what information, see Appendix 3. We draw the conclusion that we need to explore new points of view and need similarity measures that contain more information if we want them to be more effective. In addition, it is necessary to satisfy the five property conditions strictly for similarity measure, or, counter-intuitive cases are apt to occur. Vague (intuitionistic fuzzy) sets have the advantage of being able to consider waver (degree of hesitancy), but existing similarity measures have not yet considered waver. If they did so, perhaps better performance might be obtained.

Each similarity measure expression has its own measuring focus as showed in Appendix 3 although they all evaluate the similarities in fuzzy/vague sets. You can also find in Appendix 2 that the results of similarity measures are similar except some counter-intuitive cases, we summarize the conditions these measures are similar here: (1) They all focus on differences between intuitionistic fuzzy/vague sets A and B based on the comparison of interval value $[t_A \; 1 - f_A]$ and $[t_B \; 1 - f_B]$; (2) they can meet all or most of the properties condition of similarity measure in Definition 2.2.1, that is P1–P5.

Among the existing 12 similarity measures between IFSs/vague sets, it is obvious that $S_h^p(A, B)$ has no counter-intuitive cases, but we still cannot say $S_h^p(A, B)$ is the best and should replace others. On the contrary, we think all existing similarity measures are valuable. There exists two reasons behind this thought: first, a new similarity measure is proposed, always accompanying with explanations of overcoming counter-intuitive cases of other methods. In all the existing involved papers, it is really difficult to find all the counter-intuitive cases just by enumerating. Second, there are different selection criteria and requirements during specific application procedure of similarity measure. It is important that you set up your viewpoints that you focus on when you measure similarity. If you focus on waver, then S_h^p, which considers $1 - t_A(x_i) - f_A(x_i)$, might be best. Perhaps you feel $|t_A(x_i) - f_A(x_i)|$ should be the primary focus, in which case you might choose S_C and S_L, etc. In a word, different needs results in different selection, different selection results in a slightly different measurement results. In addition, we should not only consider specific needs but also pay attention to their drawbacks when we applied them into practice. Appendix 1 is good reference, contributing to helping you select a better one among existing similarity measures of IFSs/vague sets.

Similarity measures between IFSs/vague sets is a more realistic and promising way to solve some problem, we should try our best to improve them and apply them into practice.

Appendix 1. Similarity measure expressions and counter-intuitive cases

	Expressions	Counter-intuitive cases
(1)	$S_C(A,B) = 1 - \frac{\sum_{i=1}^{n} \lvert S_A(x_i) - S_B(x_i)\rvert}{2n}$ $S_A(x_i) = t_A(x_i) - f_A(x_i),\ S_B(x_i) = t_B(x_i) - f_B(x_i)$	$A = [(x,0,0)]B = [(x,0.5,0.5)] \times S_C(A,B) = 1$
(2)	$S_H(A,B) = 1 - \frac{\sum_{i=1}^{n} (\lvert t_A(x_i) - t_B(x_i)\rvert + \lvert f_A(x_i) - f_B(x_i)\rvert)}{2n}$	Type I: $A = \{(x,0.3,0.3)\},\ B = \{(x,0.4,0.4)\}$ $C = \{(x,0.3,0.4)\},\ D = \{(x,0.4,0.3)\} \times S_H(A,B) = S_H(C,D) = 0.9$ Type II: $A = \{(x,1,0)\},\ B = \{(x,0,0)\}$ $C = \{(x,0.5,0.5)\} \times S_H(A,B) = S_H(C,B) = 0.5$
(3)	$S_L(A,B) = 1 - \frac{\sum_{i=1}^{n} \lvert S_A(x_i) - S_B(x_i)\rvert)}{4n} - \frac{\sum_{i=1}^{n} \lvert t_A(x_i) - t_B(x_i)\rvert + \lvert f_A(x_i) - f_B(x_i)\rvert}{4n}$	$A = \{(x,0.4,0.2)\},\ B = \{(x,0.5,0.3)\}$ $C = \{(x,0.5,0.2)\} \times S_L(A,B) = S_L(A,C) = 0.95$
(4)	$S_O(A,B) = 1 - \sqrt{\frac{\sum_{i-1}^{n} (t_A(x_i) - t_B(x_i))^2 + (f_A(x_i) - f_B(x_i))^2}{2n}}$	Same as that of Type I of $S_H(A,B)$
(5)	$S_{DC}(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} \lvert \psi_A(x_i) - \psi_B(x_i)\rvert^p}{n}}$ $\psi_A(x_i) = \frac{t_A(x_i) + 1 - f_A(x_i)}{2} \quad \psi_B(x_i) = \frac{t_B(x_i) + 1 - f_B(x_i)}{2}$	Same as that of $S_C(A,B)$
(6)	$S_{HB}(A,B) = \frac{1}{2}(\rho_t(A,B) + \rho_f(A,B))$ $\rho_t(A,B) = S_{DC}(t_A(x_i), t_B(x_i))$ $\rho_f(A,B) = S_{DC}(1 - f_A(x_i), 1 - f_B(x_i))$	Same as that of $S_H(A,B)$
(7)	$S_e^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\phi_t(x_i) + \phi_f(x_i))^p}{n}}$ $\phi_t(x_i) = \lvert t_A(x_i) - t_B(x_i)\rvert/2$ $\phi_f(x_i) = \lvert (1 - f_A(x_i))/2 - (1 - f_B(x_i))/2\rvert$	Same as that of $S_H(A,B)$
(8)	$S_s^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\varphi_{s1}(x_i) + \varphi_{s2}(x_i))^p}{n}}$ $\varphi_{s1}(x_i) = \lvert m_{A1}(x_i) - m_{B1}(x_i)\rvert/2,$ $\varphi_{s2}(x_i) = \lvert m_{A2}(x_i) - m_{B2}(x_i)\rvert/2$ $m_{A1}(x_i) = (t_A(x_i) + m_A(x_i))/2,$ $m_{B1}(x_i) = (t_B(x_i) + m_B(x_i))/2$ $m_{A2}(x_i) = (m_A(x_i) + 1 - f_A(x_i))/2,$ $m_{B2}(x_i) = (m_B(x_i) + 1 - f_B(x_i))/2$ $m_A(x_i) = (t_A(x_i) + 1 - f_A(x_i))/2,$ $m_B(x_i) = (t_B(x_i) + 1 - f_B(x_i))/2$	Same as that of $S_L(A,B)$

(*continued on next page*)

Appendix 1 (*continued*)

	Expressions	Counter-intuitive cases
(9)	$S_h^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}(\eta_1(i)+\eta_2(i)+\eta_3(i))^p}{3n}}$ $\eta_1(i) = \phi_t(x_i) + \phi_f(x_i)$ (occurring in S_e^p) or $\eta_1(i) = \varphi_{s1}(x_i) + \varphi_{s2}(x_i)$ (occurring in S_s^p) $\eta_2(i) = \lvert\psi_A(x_i) - \psi_B(x_i)\rvert$ (occurring in S_{DC}) $\eta_3(i) = \max(l_A(i), l_B(i)) - \min(l_A(i), l_B(i))$ $l_A(i) = (1 - f_A(x_i) - t_A(x_i))/2$ $l_B(i) = (1 - f_B(x_i) - t_B(x_i))/2$	
(10)	$S_{HY}^1(A,B) = 1 - d_H(A,B)$ $d_H(A,B) = \frac{1}{n}\sum_{i=1}^{n} \max(\lvert t_A(x_i) - t_B(x_i)\rvert, \lvert f_A(x_i) - f_B(x_i)\rvert)$	Same as that of S_L and Type I counter cases of S_H $A = [(x,1,0)]$ and $B = [(x,0,0)]$ $^{**}S_{HY}(A,B) = 1$
(11)	$S_{HY}^2(A,B) = (e^{-d_H(A,B)} - e^{-1})/(1 - e^{-1})$ Same as $d_H(A,B)$	Same as that of S_L and Type I counter cases of S_H $A = [(x,1,0)]$ and $B = [(x,0,0)]$ $^{**}S_{HY}(A,B) = 1$
(12)	$S_{HY}^3(A,B) = (1 - d_H(A,B))/(1 + d_H(A,B))$ Same as $d_H(A,B)$	Same as that of S_L and Type I counter cases of S_H $A = [(x,1,0)]$ and $B = [(x,0,0)]$ $^{**}S_{HY}(A,B) = 1$

Appendix 2. Demonstration table of counter-intuitive cases (bold italic)

	1	2	3	4	5	6
$A = [(x, t_A, f_A)]$	$[(x, 0.3, 0.3)]$	$[(x, 0.3, 0.4)]$	$[(x, 1, 0)]$	$[(x, 0.5, 0.5)]$	$[(x, 0.4, 0.2)]$	$[(x, 0.4, 0.2)]$
$B = [(x, t_B, f_B)]$	$[(x, 0.4, 0.4)]$	$[(x, 0.4, 0.3)]$	$[(x, 0, 0)]$	$[(x, 0, 0)]$	$[(x, 0.5, 0.3)]$	$[(x, 0.5, 0.2)]$
S_C	1	0.9	0.5	***1***	***1***	0.95
S_H	***0.9***	***0.9***	***0.5***	***0.5***	0.9	0.95
S_L	0.95	0.9	0.5	0.75	***0.95***	***0.95***
S_O	***0.9***	***0.9***	0.3	0.5	0.9	0.93
S_{DC}	1	0.9	0.5	***1***	***1***	0.95
S_{HB}	***0.9***	***0.9***	***0.5***	***0.5***	0.9	0.95
S_e^p	***0.9***	***0.9***	***0.5***	***0.5***	0.9	0.95
S_s^p	0.95	0.9	0.5	0.75	***0.95***	***0.95***
S_h^{p*}	0.93	0.933	0.5	0.67	0.93	0.95
S_{HY}^1	***0.9***	***0.9***	***0***	0.5	***0.9***	***0.9***
S_{HY}^2	***0.85***	***0.85***	***0***	0.38	***0.85***	***0.85***
S_{HY}^3	***0.82***	***0.82***	***0***	0.33	***0.82***	***0.82***

Appendix 3. Measuring focus of similarity measure expressions

	Differences between t_A and t_B and between f_A and f_B (C1)	Differences between median values of intervals or subintervals (C2)	Difference of degree of support (C3)	Difference between length of interval (C4)
S_C			√	
S_H	√			
S_L	√		√	
S_O	√			
S_{DC}		√		
S_{HB}	√			

Appendix 3 (*continued*)

	Differences between t_A and t_B and between f_A and f_B (C1)	Differences between median values of intervals or subintervals (C2)	Difference of degree of support (C3)	Difference between length of interval (C4)
S_e^p	$\surd$			
S_s^p		$\surd$		
S_h^{p*}	$\surd$	$\surd$		$\surd$
S_{HY}^1	$\surd$			
S_{HY}^2	$\surd$			
S_{HY}^3	$\surd$			

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